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On the modular Lie superalgebra Ω

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ABSTRACT

A class of finite-dimensional simple modular Lie superalgebra Ω was constructed and its derivation superalgebra was determined in Zhang and Zhang (2009) [24]. The purpose of the present paper is to continue the investigation of finite-dimensional modular Lie superalgebra Ω. A nonnatural filtration of finite-dimensional modular Lie superalgebra Ω over a field of positive characteristic is determined. It is proved that the nonnatural filtration of Ω is invariant under the automorphism group of Ω. Moreover, we obtain the sufficient and necessary conditions of $\Omega(r, m, q, \underline{s}, H) \cong \Omega(r', m', q', \underline{s}', H')$, i.e., we classified the Lie superalgebra of Ω type.

1. Introduction

The theory of Lie superalgebras over a field of characteristic zero has experienced a rather vigorous development in mathematics throughout the last fifty years (see [5–7,11]). But the research on modular Lie superalgebras, i.e., Lie superalgebras over a field of prime characteristic, just began in recent years. Many important results of modular Lie superalgebras have also been obtained (see [2,9,10,12,20]). However, the complete classification of the finite-dimensional simple modular Lie superalgebras remains an open problem. So constructing new finite-dimensional simple modular Lie algebras and studying its derivation algebra, filtration structure as well as other natural properties are of great importance before the problem is resolved.

It is well known that filtration structures play a predominant role both in the classification of modular Lie algebras (see [1,4,8,13,15,19]) and nonmodular Lie superalgebras (see [6,11]). Similarly, filtration structures will play an important role in the classification of modular Lie superalgebras. The filtration structures of modular Lie algebras of Cartan type and nonmodular Lie superalgebras were studied in [8,14,16,5], respectively. The natural filtrations of modular Lie superalgebras W and S were proved to be invariant by using ad-nilpotent elements in paper [21]; the same results for modular Lie superalgebras H and K were obtained by means of a minimal dimension of image spaces (see [22,23]). The invariance of the nontrivial transitive filtration of modular Lie superalgebra HO was investigated in paper [18].

This paper is devoted to the study of the filtration structure of the Lie superalgebra Ω by the method of minimal dimension of image spaces. This paper is organized as follows. In Section 2 we first simply describe the Lie superalgebra Ω and then review certain known results in [24]. In Section 3 a nonnatural filtration of modular Lie superalgebra Ω is given and the invariance of the filtration of Ω is proved. Furthermore, we obtain that integers r, m and q in the definition of modular Lie superalgebra Ω are intrinsic.

2. Preliminaries

Throughout this article, $\mathbb{F}$ denotes an algebraic closed field of characteristic $p > 3$ and $\mathbb{F}$ is not equal to its prime field Π. Let, for $m > 0$, $E = \{z_1, \ldots, z_m\} \in \mathbb{F}$ be linearly independent over the prime field Π and the additive subgroup H generated

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by E does not contain 1. If $\lambda \in H$, then we let $\lambda = \sum_{i=1}^{m} \lambda_i z_i$ and $y^\lambda = y_1^{\lambda_1} \cdots y_m^{\lambda_m}$, where $0 \le \lambda_i < p$. We use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ for the set of non-negative integers. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the ring of integers modulo 2. Given $n \in \mathbb{N}$ and $r = 2n+2$, we put $M = \{1, \ldots, r-1\}$. Suppose that $\mu_1, \cdots, \mu_{r-1} \in \mathbb{F}$ such that $\mu_1 = 0$, $\mu_j + \mu_{n+j} = 1$, $j = 2, \ldots, n+1$. Let $\underline{s} = (s_1+1, \ldots, s_{r-1}+1) \in \mathbb{N}^{r-1}$, where $s_i \in \mathbb{N}_0$, $i \in M$. We define a truncated polynomial algebra

$$A = \mathbb{F}[x_{10}, x_{11}, \ldots, x_{1s_1}, \ldots, x_{r-10}, x_{r-11}, \ldots, x_{r-1s_{r-1}}, y_1, \ldots, y_m]$$

such that

$$x_{ij}^p = 0, \quad \forall i \in M, j = 0, 1, \ldots, s_i; \qquad y_i^p = 1, \quad i = 1, \ldots, m.$$

Let $\pi_i = p^{s_i+1} - 1$, $i \in M$. If $k_i \in \mathbb{N}_0$ is such that $0 \le k_i \le \pi_i$, then k_i can be uniquely expressed in p-adic form: $k_i = \sum_{v=0}^{s_i} \varepsilon_v(k_i) p^v$, where $0 \le \varepsilon_v(k_i) < p$. We set $x_i^{k_i} = \prod_{v=0}^{s_i} x_{iv}^{\varepsilon_v(k_i)}$. For $0 \le k_i, k_i' \le \pi_i$ and $x_i^{k_i'} = \prod_{v=0}^{s_i} x_{iv}^{\varepsilon_v(k_i')}$, it is easy to see that

$$x_i^{k_i} x_i^{k_i'} = x_i^{k_i+k_i'} \ne 0 \Leftrightarrow \varepsilon_v(k_i) + \varepsilon_v(k_i') < p, \quad v = 0, 1, \ldots, s_i. \tag{1}$$

Let $Q = \{(k_1, \ldots, k_{r-1}) \mid 0 \le k_i \le \pi_i, i \in M\}$. If $k = (k_1, \ldots, k_{r-1}) \in Q$, we write $x^k = x_1^{k_1} \cdots x_{r-1}^{k_{r-1}}$.

Let $\Lambda(q)$ be the Grassmann superalgebra over $\mathbb{F}$ in q variables $\xi_{r+1}, \ldots, \xi_{r+q}$, where $q \in \mathbb{N}$ and $q > 1$. Denote the tensor product by $\widetilde{\Omega} := A \otimes_{\mathbb{F}} \Lambda(q)$. Obviously, $\widetilde{\Omega}$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of A and the natural $\mathbb{Z}_2$-gradation of $\Lambda(q)$:

$$\widetilde{\Omega}_{\bar{0}} = A \otimes_{\mathbb{F}} \Lambda(q)_{\bar{0}}, \qquad \widetilde{\Omega}_{\bar{1}} = A \otimes_{\mathbb{F}} \Lambda(q)_{\bar{1}}.$$

If $f \in A$, $g \in \Lambda(q)$, then we abbreviate $f \otimes g$ to fg. For $k \in \{1, \ldots, q\}$, we set

$$\mathbb{B}_k = \{\langle i_1, i_2, \ldots, i_k \rangle \mid r+1 \le i_1 < i_2 < \cdots < i_k \le r+q\}$$

and $\mathbb{B}(q) = \bigcup_{k=0}^{q} \mathbb{B}_k$, where $\mathbb{B}_0 = \varnothing$. If $u = \langle i_1, \ldots, i_k \rangle \in \mathbb{B}_k$, we let $|u| = k$, $\{u\} = \{i_1, \ldots, i_k\}$ and $\xi^u = \xi_{i_1} \cdots \xi_{i_k}$. Put $|\varnothing| = 0$ and $\xi^\varnothing = 1$. Then $\{x^k y^\lambda \xi^u \mid k \in Q, \lambda \in H, u \in \mathbb{B}(q)\}$ is an $\mathbb{F}$-basis of $\widetilde{\Omega}$.

If L is a Lie superalgebra, then $h(L)$ denotes the set of all $\mathbb{Z}_2$-homogeneous elements of L, i.e., $h(L) = L_{\bar{0}} \cup L_{\bar{1}}$. If $|x|$ appears in some expression in this paper, we always regard x as a $\mathbb{Z}_2$-homogeneous element and $|x|$ as the $\mathbb{Z}_2$-degree of x.

Set $s = r+q$, $T = \{r+1, \ldots, s\}$, and $R = M \cup T$. Put $M_1 = \{2, \ldots, r-1\}$. Define $\tilde{i} = \bar{0}$, if $i \in M_1$, and $\tilde{i} = \bar{1}$, if $i \in T$. Let

$$i' = \begin{cases} i+n, & 2 \le i \le n+1 \\ i-n, & n+2 \le i \le r-1 \\ i, & r+1 \le i \le s, \end{cases} \qquad [i] = \begin{cases} 1, & 2 \le i \le n+1 \\ -1, & n+2 \le i \le r-1 \\ 1, & r+1 \le i \le s. \end{cases}$$

Put $e_i = (\delta_{i1}, \ldots, \delta_{ir-1})$, $i = 1, \ldots, r-1$. Let D_i, $i \in R$, be the linear transformations of $\widetilde{\Omega}$ such that

$$D_i(x^k y^\lambda \xi^u) = \begin{cases} k_i^* x^{k-e_i} y^\lambda \xi^u, & i \in M \\ \partial/\partial\xi_i, & i \in T, \end{cases}$$

where k_i^* is the first nonzero number of $\varepsilon_0(k_i), \varepsilon_1(k_i), \ldots, \varepsilon_{s_i}(k_i)$. Then $D_i \in \operatorname{Der}\widetilde{\Omega}$. Set

$$\bar{\partial} = I - \sum_{j \in M_1} \mu_j x_{j0} \frac{\partial}{\partial x_{j0}} - \sum_{j=1}^{m} z_j y_j \frac{\partial}{\partial y_j} - 2^{-1} \sum_{j \in T} \xi_j \frac{\partial}{\partial \xi_j},$$

where I is the identity mapping of $\widetilde{\Omega}$. For $f \in h(\widetilde{\Omega})$, $g \in \widetilde{\Omega}$, we define a bilinear operation $[\ ,\ ]$ in $\widetilde{\Omega}$ such that

$$[f, g] = D_1(f)\bar{\partial}(g) - \bar{\partial}(f)D_1(g) + \sum_{i \in M_1 \cup T} [i](-1)^{\tilde{i}|f|} D_i(f) D_{i'}(g).$$

Then $\widetilde{\Omega}$ becomes a simple Lie superalgebra for the operation $[\ ,\ ]$ defined above.

Let $x_i = x_i^1 = x_{i0}$, $\forall i \in M$. Set $\pi = (\pi_1, \ldots, \pi_{r-1}) \in Q$, $\omega = \langle r+1, \ldots, s \rangle \in \mathbb{B}(q)$.

By computation, we obtain the commutator subalgebra of $\widetilde{\Omega}$:

$$[\widetilde{\Omega}, \widetilde{\Omega}] = \{x^k y^\lambda \xi^u \mid (k, \lambda, u) \ne (\pi, n+2-2^{-1}q, \omega)\}.$$

Define $\Omega := [\widetilde{\Omega}, \widetilde{\Omega}]$. In some cases, we denote Ω by $\Omega(r, m, q, \underline{s}, H)$ in detail and call $\Omega(r, m, q, \underline{s}, H)$ the Lie superalgebra of Ω type.

If $2n + 4 - q \not\equiv 0 \pmod p$, then $\lambda + 2^{-1}q - n - 2 \ne 0$. It is easily seen that $\Omega = \widetilde{\Omega}$.

Now we give a $\mathbb{Z}$-gradation of Ω: $\Omega = \bigoplus_{i \in X} \Omega_i$, where

$$\Omega_i = \langle x^k y^\lambda \xi^u \mid \sum_{i \in M_1} k_i + 2k_1 + |u| - 2 = i \rangle,$$

and $X = \{-2, -1, \ldots, \tau\}$, $\tau = \sum_{i\in M_1} \pi_i + 2\pi_1 + q - 2$. Let $f \in \Omega$. If $f \in \Omega_i$, then f is called a $\mathbb{Z}$-homogeneous element and i is the $\mathbb{Z}$-degree of f which is denoted by $\mathrm{zd}(f)$.

Set $\mathrm{Der}\,\Omega := \bigoplus_{\alpha\in\mathbb{Z}_2} \mathrm{Der}_\alpha \Omega$, where $\mathrm{Der}_\alpha\Omega$ denotes the linear space of all derivations of degree α of Ω, i.e.,

$$\mathrm{Der}_\alpha\Omega = \{\varphi \in \mathrm{Der}\,\Omega \mid \varphi(\Omega_\beta) \subseteq \Omega_{\alpha+\beta}, \forall\, \beta \in \mathbb{Z}_2\}.$$

We know that $\mathrm{Der}\,\Omega$ is also a $\mathbb{Z}$-graded Lie superalgebra:

$$\mathrm{Der}\,\Omega = \bigoplus_{t\in Y} \mathrm{Der}_t\Omega, \qquad \mathrm{Der}_t\Omega = \{\varphi \in \mathrm{Der}\,\Omega \mid \varphi(\Omega_i) \subseteq \Omega_{i+t}, \forall\, i \in \mathbb{Z}\},$$

where $Y = \{-\zeta, -\zeta + 1, \ldots, \zeta\}$ and $\zeta = \tau + 2$.

Let $\Delta = \{\theta : H \to \mathbb{F} \mid \theta(\lambda + \eta) = \theta(\lambda) + \theta(\eta), \forall\, \lambda, \eta \in H\}$. For $\theta \in \Delta$, we define a linear transformation D_θ of Ω such that $D_\theta(x^k y^\lambda \xi^u) = \theta(\lambda)x^k y^\lambda \xi^u$. Clearly $D_\theta \in \mathrm{Der}\,\Omega$.

Put $W_1 = \{D_\theta \mid \theta \in \Delta\}$. Then W_1 is m-dimensional linear space. Set $W_2 = \langle D_i^{p^{v_i}} \mid \forall\, i \in M, 0 < v_i \le s_i\rangle$.

Proposition 2.1 (*[24]*). $\mathrm{Der}\,\Omega = \mathrm{ad}\,\Omega \oplus W_1 \oplus W_2$.

Lemma 2.2 (*[24]*). *Let $f \in \Omega$. If $D_i(f) = 0$, $\forall\, i \in R$, then $f = \sum_{\lambda\in H} a_\lambda y^\lambda$, where $a_\lambda \in \mathbb{F}$.*

If $2n + 4 - q \equiv 0 \pmod p$, then $\Omega = \{x^k y^\lambda \xi^u \mid (k, \lambda, u) \neq (\pi, 0, \omega)\}$. In this case we write Ω^* for Ω.

Proposition 2.3 (*[24]*). $\mathrm{Der}\,\Omega^* = \mathrm{ad}\,\Omega^* \oplus \langle \mathrm{ad}\, x^\pi \xi^\omega\rangle \oplus W_1 \oplus W_2$.

3. Filtration

In this section, we first give a nonnatural filtration of Ω and then introduce some technical Lemmas which will be used to determine the invariance of the filtration. Finally, an intrinsic property (see, Theorem 3.11) is proved by the invariance of the filtration; that is, the integers in the definition of modular Lie superalgebra Ω are intrinsic. Therefore, we classify the modular Lie superalgebra of Ω type in the sense of isomorphism.

Put $I(\varphi) = \dim(\mathrm{Im}\varphi)$, where $\varphi \in \mathrm{Der}\,\Omega$. Let Γ be a set of $\mathrm{Der}\,\Omega$ and $I(\Gamma) := \min\{I(\varphi) \mid 0 \neq \varphi \in \Gamma\}$. Set

$$\chi(y) = \sum_{\eta\in H} y^\eta, \qquad b = x^\pi \xi^\omega \chi(y), \qquad B = \mathrm{ad}\, b\,|_\Omega\ .$$

If $\alpha := \{\alpha_\lambda \mid \lambda \in H\}$ is a subset of $\mathbb{F}$, then we let

$$\alpha(y) = \sum_{\lambda\in H} \alpha_\lambda y^\lambda, \qquad P_t(\alpha) = \sum_{\lambda\in H} (1-\lambda)^t \alpha_\lambda, \quad t = 0, 1.$$

Lemma 3.1. *The following statements hold.*
(1) $\mathfrak{C} := \ker B = \langle x_\xi y^\lambda \mid d_{x_\xi} \geq 2\rangle \oplus \langle x_\xi \alpha(y) \mid d_{x_\xi} = 1, P_0(\alpha) = 0\rangle \oplus \langle \alpha(y) \mid P_1(\alpha) = 0\rangle$, where $x_\xi = x^k \xi^u$ and $d_{x_\xi} = \sum_{i\in M} k_i + |u|$.
(2) $I(B) = s$, where $s = r + q$.

Proof. (1) Clearly $B(x_\xi y^\lambda) = 0$, where $d_{x_\xi} \geq 2$. Note that $\chi(y)y^\lambda = \chi(y), \forall\, \lambda \in H$. As $P_0(\alpha) = \sum_{\lambda\in H} \alpha_\lambda = 0$, by direct computation we obtain

$$B(x_1\alpha(y)) = \left(\sum_{\lambda\in H} \alpha_\lambda\right)\left(x^\pi \xi^\omega \left(\sum_{\eta\in H} \eta y^\eta\right) - (n + 2 - 2^{-1}q)b\right) = 0.$$

Similarly,

$$B(x_i\alpha(y)) = -[i']\left(\sum_{\lambda\in H} \alpha_\lambda\right) x^{\pi - e_{i'}} \xi^\omega \chi(y) = 0, \quad \forall\, i \in M_1,$$

$$B(\xi_j\alpha(y)) = (-1)^{|q|}\left(\sum_{\lambda\in H} \alpha_\lambda\right) x^\pi \xi^{\omega - \langle j\rangle} \chi(y) = 0, \quad \forall\, j \in T.$$

Since $P_1(\alpha) = \sum_{\lambda\in H} (1-\lambda)\alpha_\lambda = 0$, we have

$$B(\alpha(y)) = \left(\sum_{\lambda\in H} (\lambda - 1)\alpha_\lambda\right) x^{\pi - e_1} \xi^\omega \chi(y) = 0.$$

(2) We see that

$$B(x_1 y^\lambda) = x^\pi \xi^\omega \left(\sum_{\eta\in H} \eta y^\eta\right) - (n + 2 - 2^{-1}q)b \neq 0,$$

which is independent of λ. Similarly,

$$B(x_iy^\lambda) = -[i']x^{\pi-e_{i'}}\xi^\omega\chi(y) \neq 0, \quad \forall i \in M_1,$$
$$B(\xi_jy^\lambda) = (-1)^{|q|}x^\pi\xi^{\omega-\langle j\rangle}\chi(y) \neq 0, \quad \forall j \in T,$$
$$B(y^\lambda) = (\lambda-1)x^{\pi-e_1}\xi^\omega\chi(y) \neq 0.$$

Let $\mathfrak{N} = \langle 1, x_i, \xi_j \mid \forall i \in M, \forall\lambda \in H, \forall j \in T\rangle$. Then $\Omega = \mathfrak{C} \oplus \mathfrak{N}$. It is easy to see that $B1, Bx_i, B\xi_j, \forall i \in M, \forall j \in T$, are linearly independent. Hence $I(B) = r + q = s$, as desired. □

Let $\Omega_{(0)} = \mathfrak{C}$, $\Omega_{(-1)} = \Omega$ and define

$$\Omega_{(i)} = \{f \in \Omega_{(i-1)} \mid [f, \Omega_{(-1)}] \subseteq \Omega_{(i-1)}\}, \quad \forall i \geq 1. \tag{2}$$

We obtain a descending filtration of Ω: $\{\Omega_{(i)} \mid i \geq -1\}$.

Lemma 3.2. *If f is a nonzero homogeneous element of Ω, $f \notin \langle x^\pi\xi^\omega\alpha(y)\rangle$, then there exist two basis elements f_1, f_2 with* $\mathrm{zd}(f_i) \geq 0$, $i = 1, 2$, *such that $[f, f_1]$ and $[f, f_2]$ are linearly independent.*

Proof. (1) Let $D_j(f) = 0$, $\forall j \in T$. Then every term of f can be expressed in $\alpha_{k\lambda}x^ky^\lambda$ form, where $\alpha_{k\lambda} \in \mathbb{F}$, and two cases arise:

Case 1: $\mathrm{zd}(f) = |\pi| - 2$. We can suppose that $f = \sum_{\lambda\in S}\alpha_{\pi\lambda}x^\pi y^\lambda$, where $0 \neq \alpha_{\pi\lambda} \in \mathbb{F}$ and $S \subseteq H$. So we get

$$[f, x_i\xi_j] = -[i']\sum_{\lambda\in S}\alpha_{\pi\lambda}x^{\pi-e_{i'}}y^\lambda\xi_j \neq 0,$$
$$[f, x_{i'}\xi_j] = -[i]\sum_{\lambda\in S}\alpha_{\pi\lambda}x^{\pi-e_i}y^\lambda\xi_j \neq 0,$$

and they are linearly independent.

Case 2: $\mathrm{zd}(f) < |\pi| - 2$. We may assume that $f = \sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}x^ky^\lambda$, where $\Delta \subseteq Q$, $S \subseteq H$, $\alpha_{k\lambda} \in \mathbb{F}$, and $\alpha_{k\lambda} \neq 0$. Put $\beta_{k\lambda} = 1 - \lambda - \sum_{i\in M_1}k_i\mu_i$. For $i, j \in T$ and $i \neq j$, we have

$$z_1 := \left[\sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}x^ky^\lambda, x_1\right] = \sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}(k_1^*x^{k-e_1}x_1y^\lambda - \beta_{k\lambda}x^ky^\lambda),$$
$$z_2 := \left[\sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}x^ky^\lambda, x_1\xi_i\right] = \sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}(2^{-1}k_1^*x^{k-e_1}x_1y^\lambda\xi_i - \beta_{k\lambda}x^ky^\lambda\xi_i),$$
$$z_3 := \left[\sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}x^ky^\lambda, x_1\xi_i\xi_j\right] = \sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}(-\beta_{k\lambda}x^ky^\lambda\xi_i\xi_j).$$

If there is $k \in \Delta$ such that $\varepsilon_0(k_1) \neq 0$, then $\varepsilon_v(k_1 - 1) + \varepsilon_v(1) < p$ for any $v \geq 0$. Equality (1) ensures that $x^{k-e_1}x_1 = x^k$. Similarly, $\varepsilon_0(k_1) = 0$ implies that $\varepsilon_0(k_1 - 1) + \varepsilon_0(1) = p$ and thereby $x^{k-e_1}x_1 = 0$. Put $W = \{k \in \Delta \mid \varepsilon_0(k_1) \neq 0\}$. Thus

$$z_1 = \sum_{k\in W,\lambda\in S}\alpha_{k\lambda}(k_1^* - \beta_{k\lambda})x^ky^\lambda + \sum_{k\in\Delta\setminus W,\lambda\in S}\alpha_{k\lambda}(-\beta_{k\lambda}x^ky^\lambda),$$
$$z_2 = \sum_{k\in W,\lambda\in S}\alpha_{k\lambda}(2^{-1}k_1^* - \beta_{k\lambda})x^ky^\lambda\xi_i + \sum_{k\in\Delta\setminus W,\lambda\in S}\alpha_{k\lambda}(-\beta_{k\lambda}x^ky^\lambda\xi_i),$$
$$z_3 = \sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}(-\beta_{k\lambda}x^ky^\lambda\xi_i\xi_j).$$

If there is 2-tuple (k, λ), $k \in \Delta$, $\lambda \in S$, such that $\beta_{k\lambda} \not\equiv 0 \pmod p$, then at least two of elements z_1, z_2, z_3 are nonzero and our assertion is affirmed. Otherwise, z_1 and z_2 are linearly independent.

If $\varepsilon_0(k_1) = 0$ for all $k \in \Delta$, then $x^{k-e_1}x_1 = 0$ ensures that

$$z_1 = \sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}(-\beta_{k\lambda}x^ky^\lambda),$$
$$z_2 = \sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}(-\beta_{k\lambda}x^ky^\lambda\xi_i),$$
$$z_3 = \sum_{k\in\Delta,\lambda\in S}\alpha_{k\lambda}(-\beta_{k\lambda}x^ky^\lambda\xi_i\xi_j).$$

If there exists the 2-tuple (k, λ), $k \in \Delta$, $\lambda \in S$, such that $\beta_{k\lambda} \not\equiv 0 \pmod p$, then z_1, z_2 and z_3 are all nonzero elements. Considering the basic elements x^ky^λ, $x^ky^\lambda\xi_i$ and $x^ky^\lambda\xi_i\xi_j$ on the right-hand side, we know that any two of elements z_1, z_2, z_3

are linearly independent. If $\beta_{k\lambda} \equiv 0 \pmod p$, $\forall\, k \in \Delta$, $\forall\, \lambda \in S$, then for any $k \in \Delta$, there is $i \in M_1$ such that $k_i \neq 0$. For $j \in T$, we have

$$[f, x_1 x_{i'}] = [i]\alpha_{k\lambda} k_i^* x^{k-e_i} y^\lambda x_1 + \cdots \neq 0, \qquad [f, x_{i'}\xi_j] = [i]\alpha_{k\lambda} k_i^* x^{k-e_i} y^\lambda \xi_j + \cdots \neq 0.$$

Since their $\mathbb{Z}$-degrees are unequal, $[f, x_1 x_{i'}]$ and $[f, x_{i'}\xi_j]$ are linearly independent.

(2) If there is $l \in T$ such that $D_l(f) \neq 0$, then f has only two cases.

(a) $D_i^{\pi_i}(f) \neq 0$, $\forall\, i \in M$. Since $f \notin \langle x^\pi \xi^\omega \alpha(y)\rangle$, there exists $j \in T$ such that $D_j(f) = 0$. So we can suppose that $f = x^\pi y^\lambda \xi^u + \cdots$, where $u \neq \varnothing$, $u \neq \omega$ and $j \notin \{u\}$. Then

$$z_1 := [f, x_i\xi_j] = -[i']x^{\pi-e_{i'}} y^\lambda \xi^u \xi_j + \cdots \neq 0,$$
$$z_2 := [f, x_{i'}\xi_j] = -[i]x^{\pi-e_i} y^\lambda \xi^u \xi_j + \cdots \neq 0.$$

It is easy to see that z_1 and z_2 are linearly independent.

(b) There is $i \in M$ such that $D_i^{\pi_i}(f) = 0$. If $D_j(f) \neq 0$, $\forall\, j \in T$, then we may assume that $f = x^k y^\lambda \xi^\omega + \cdots$, where $k_i \neq \pi_i$, $i \in M$. Hence there exists t $(0 \leq t \leq s_i)$ such that $x^k x^{p^t e_i} \neq 0$. Then

$$z_1 := [f, x^{p^t e_i}\xi_j] = (-1)^{|q|} x^k x^{p^t e_i} y^\lambda \xi^{\omega-\langle j\rangle} + \cdots \neq 0,$$
$$z_2 := [f, x^{p^t e_i}\xi_{j+1}] = (-1)^{|q|} x^k x^{p^t e_i} y^\lambda \xi^{\omega-\langle j+1\rangle} + \cdots \neq 0,$$

and they are linearly independent.

If there exists $j \in T$ such that $D_j(f) = 0$, then let

$$f = x^k y^\lambda \xi^u + \sum_{l,\eta,v} a_{l\eta v} x^l y^\eta \xi^v,$$

where $a_{l\eta v} \in \mathbb{F}$, $u \neq \varnothing$. By $D_j(f) = 0$, we see that $j \notin \{u\}, j \notin \{v\}$. According to $D_i^{\pi_i}(f) = 0$, we get $k_i < \pi_i$ and $l_i < \pi_i$. Now let $\iota \in \{u\}$. Then

$$z_1 := [f, \xi_\iota \xi_j] = (-1)^{|u|} x^k y^\lambda \xi^{u-\langle\iota\rangle}\xi_j + \cdots \neq 0.$$

By virtue of $k_i < \pi_i$, there is $t \in \{0, 1, \ldots, s_i\}$ such that $x^k x^{p^t e_i} \neq 0$. Then

$$z_2 := [f, x^{p^t e_i}\xi_\iota] = (-1)^{|u|} x^k x^{p^t e_i} y^\lambda \xi^{u-\langle\iota\rangle} + \cdots \neq 0$$

and our assertion is affirmed. □

We denote by $\varepsilon(f)$ the nonzero $\mathbb{Z}$-homogeneous part of $f \in \Omega$ with the least $\mathbb{Z}$-degree.

Lemma 3.3 ([22]). *Let $f_1, \ldots, f_t \in \Omega\backslash\{0\}$. If $\{f_i \mid i = 1, \ldots, t\}$ are linearly dependent, then $\{\varepsilon(f_i) \mid i = 1, \ldots, t\}$ are linearly dependent.*

Lemma 3.4. *Let $f \in h(\Omega)$ and $f \notin \langle x^\pi \xi^\omega \alpha(y)\rangle$. Then $I(\mathrm{ad} f) > s$.*

Proof. According to Lemma 3.3, we can suppose that f is a $\mathbb{Z}$-homogeneous element. We shall proceed in two steps.

(i) $[f, 1] = 0$. Then we have $D_1(f) = 0$. Let

$$R_1 = \big\{i \mid i \in M_1,\ [f, x_i] = 0\big\}, \qquad R_2 = \big\{j \mid j \in T,\ [f, \xi_j] = 0\big\}.$$

(a) If $R_1 \cup R_2 = M_1 \cup T$, then $D_i(f) = 0$, $\forall\, i \in R = M \cup T$. By Lemma 2.2, we may assume that $f = y^\lambda$, $\lambda \in H$. Then

$$[f, x^k \xi^u] = [y^\lambda, x^k \xi^u] = k_1^*(\lambda - 1) x^{k-e_1} y^\lambda \xi^u.$$

Hence $I(\mathrm{ad} f) \geq (p^{s_1+1} - 1)p^{\sum_{i\in M_1}(s_i+1)+m} 2^q \geq (p-1)p^{r-1}2^q > r + q = s$.

(b) Let $R_2 = \varnothing$, $|R_1| \leq 1$. If $|R_1| = 0$, i.e., $R_1 = \varnothing$, then $\{[f, x_i], [f, \xi_j] \mid i \in M_1, j \in T\}$ are linearly independent. If $|R_1| = 1$, we let $R_1 = \{l\}$. Obviously $\{[f, x_i], [f, \xi_j] \mid i \in M_1\backslash\{l\}, j \in T\}$ are linearly independent.

The assumption $D_1(f) = 0$ implies that $D_1(\bar\partial(f)) = 0$; that is, x_1 does not arise in $\bar\partial(f)$. Hence if $\bar\partial(f) \neq 0$, then $x_1^{k_1}\bar\partial(f) \neq 0$, $0 \leq k_1 \leq \pi_1$. Therefore

$$[f, x_1^{k_1} y^\lambda] = -k_1^* x_1^{k_1-1} y^\lambda \bar\partial(f) \neq 0, \quad 1 \leq k_1 \leq \pi_1.$$

If $\bar\partial(f) = 0$, similarly, by $D_j(f) \neq 0$, $\forall\, j \in T$, we have

$$[f, x_1^{k_1}\xi_j y^\lambda] = (-1)^{|f|} D_j(f) x_1^{k_1} y^\lambda \neq 0, \quad 0 \leq k_1 \leq \pi_1.$$

Thus $I(\mathrm{ad} f) \geq (r + q - 3) + \pi_1 p^m \geq (r + q - 3) + p(p-1) > s$.

(c) Let $\varnothing \neq R_1 \cup R_2 \neq M_1 \cup T$. Set $J' = \{i \in R_1 \mid i' \in R_1\}$. So we may assume that $J' = \{i_1, i_1', \ldots, i_u, i_u'\}$. Put $J_1 = R_1 \backslash J' = \{i_{u+1}, \ldots, i_{u+t}\}$ and $R_2 = \{j_1, \ldots, j_h\}$. Let $J_2 = \{i'_{u+1}, \ldots, i'_{u+t}\}$ and $\bar{J} = (M_1 \cup T) \backslash (R_1 \cup R_2 \cup J_2)$. Put

$$x^\gamma = \prod_{k \in J'} x^{\gamma_k e_k}, \quad \gamma_k = 0, 1, \ldots, \pi_k, \qquad \xi^\upsilon = \prod_{j \in R_2} \xi_j^{\upsilon_j}, \quad \upsilon_j = 0, 1.$$

For any $l' \in J_2$ and $\beta_{l'} \in \{1, 2, \ldots, p-1\}$, we see that

$$[f, x^\gamma x^{\beta_{l'} e_{l'}} \xi^\upsilon] = [l] \beta_{l'} D_l(f) x^\gamma x^{\beta_{l'} e_{l'} - e_{l'}} \xi^\upsilon. \tag{3}$$

For all $j \in \bar{J}$, we obtain

$$[f, x^\gamma x_j \xi^\upsilon] = [j'] D_{j'}(f) x^\gamma \xi^\upsilon, \tag{4}$$

$$[f, x^\gamma \xi^\upsilon \xi_j] = (-1)^{|f|} D_j(f) x^\gamma \xi^\upsilon. \tag{5}$$

Since $l' \in J_2, D_l(f) \neq 0$. As $D_i(f) = 0, \forall\, i \in J', f$ does not contain x_i for all $i \in J'$ and so is $D_l(f)$, which implies that $D_l(f) x^\gamma \neq 0$. By a similar argument we obtain $D_l(f) x^\gamma \xi^\upsilon \neq 0$ and so $D_l(f) x^\gamma \xi^\upsilon x^{\beta_{l'} e_{l'} - e_{l'}} \neq 0$. Similarly, $D_{j'}(f) x^\gamma \xi^\upsilon \neq 0$ and $D_j(f) x^\gamma \xi^\upsilon \neq 0$. It is easy to see that the nonzero elements on the right side of equalities (3)–(5) are linearly independent. Therefore

$$\begin{aligned} I(\operatorname{ad} f) &\geq p^{\sum_{i \in J'} (s_i + 1)} 2^h (p-1) t + p^{\sum_{i \in J'} (s_i + 1)} 2^h (s - 2 - 2u - 2t - h) \\ &\geq p^{2u} 2^h (p-1) t + p^{2u} 2^h \big(s - 2 - 2u - 2t - h\big) \\ &= p^{2u} 2^h \big(s - 2 - 2u - h + (p-3) t\big). \end{aligned}$$

Let $2u + h > 0$. If $t > 0$, by $s = r + q \geq 2n + 2 + 2 \geq 6$, we have

$$\begin{aligned} I(\operatorname{ad} f) &\geq 2^{2u+h} \big(s - 2 - (2u + h) + (p-3) t\big) \\ &= 2^{2u+h} \big(s - (2u + h)\big) + 2^{2u+h} \big((p-3) t - 2\big) \\ &\geq 2(s-1) > s. \end{aligned}$$

If $t = 0$, then $s \geq 6$ implies that

$$\begin{aligned} I(\operatorname{ad} f) &\geq p^{2u} 2^h \big(s - 2 - (2u + h)\big) \\ &= \big((p-2) + 2\big)^{2u} 2^h \big(s - 2 - (2u + h)\big) \\ &\geq (p-2)^{2u} 2^h \big(s - 2 - (2u + h)\big) + 2^{2u+h} \big(s - 2 - (2u + h)\big) \\ &\geq 2\Big(2^{2u+h} \big(s - 2 - (2u + h)\big)\Big) \geq 2\big(2(s-3)\big) = s + (3s - 12) > s. \end{aligned}$$

Let $2u + h = 0$. Then $u = h = 0$. As $R_1 \cup R_2 \neq \varnothing$, $t > 0$. If $t > 1$, then $I(\operatorname{ad} f) \geq (s-2) + (p-3) t \geq s - 2 + 4 > s$. If $t = 1$, we see that $R_2 = \varnothing$ and $|R_1| = 1$. Part (b) then yields $I(\operatorname{ad} f) > s$.

(ii) $[f, 1] \neq 0$. If there is a $j \in T$ such that $[f, \xi_j] = 0$, then

$$0 \neq [f, 1] = -\big[f, [\xi_j, \xi_j]\big] = -\big[[f, \xi_j], \xi_j\big] - (-1)^{|f|} \big[\xi_j, [f, \xi_j]\big] = 0,$$

a contradiction. So $[f, \xi_j] \neq 0, \forall j \in T$. Set $R_1 = \big\{i \in M_1 \mid [f, x_i] = 0\big\}$.

(a) $R_1 \neq \varnothing$. Let $i \in R_1$. If $i' \in R_1$, then

$$[i][f, 1] = \big[f, [x_i, x_{i'}]\big] = \big[[f, x_i], x_{i'}\big] + \big[x_i, [f, x_{i'}]\big] = 0,$$

contradicting $[f, 1] \neq 0$. Hence $i' \notin R_1$. Thus we may assume that $R_1 = \{2, \ldots, t\}$. Put $J = \{i, i' \mid i = 2, \ldots, t\}, \widetilde{J} = (M_1 \cup T) \backslash J$. Set

$$P = \{k_2 e_{2'} + \cdots + k_t e_{t'} \mid 0 \leq k_i \leq p-1, i = 2, \ldots, t\}.$$

For all $g \in \operatorname{span}_{\mathbb{F}}\{x^k \mid k \in P\}$, we shall prove: if $[f, g] = 0$, then $g = 0$. Otherwise, if $g \neq 0$, we choose $g \in \operatorname{span}_{\mathbb{F}}\{x^k \mid k \in P\}$ with the least $\mathbb{Z}$-degree satisfying $[f, g] = 0$. If $\operatorname{zd}(g) = -2$, we let $g = 1$. Then $[f, 1] = 0$, a contradiction. Let $\operatorname{zd}(g) > -2$, then there is a $i \in \{2, \ldots, t\}$ such that $D_{i'}(g) \neq 0$. Hence

$$0 = [i][x_i, [f, g]] = [i][[x_i, f], g] + [i][f, [x_i, g]] = [i][f, [x_i, g]] = [f, D_{i'}(g)].$$

It contradicts the choice of g with the least $\mathbb{Z}$-degree and the assertion follows. It is easy to see that $[f, x_j] \neq 0$, $[f, \xi_j] \neq 0$, $\forall\, j \in \widetilde{J}$. Because $|P| = p^{t-1}$, $|\widetilde{J}| = s - 2 - 2(t-1)$, $t - 1 > 0$,

$$\begin{aligned} I(\operatorname{ad} f) &\geq p^{t-1} + (s - 2 - 2(t-1)) \\ &\geq 1 + (t-1)(p-1) + (s - 2 - 2(t-1)) \\ &= s - 1 + (p-3)(t-1) > s. \end{aligned}$$

(b) $R_1 = \varnothing$. Then $[f, x_i] \neq 0,\ \forall i \in M_1$. Moreover $[f, \xi_j] \neq 0,\ \forall j \in T$. By virtue of Lemma 3.2, there exist two basis elements f_1 and f_2, $\mathrm{zd}(f_j) \geq 0,\ j = 1, 2$, such that $[f, f_1]$ and $[f, f_2]$ are linearly independent. Therefore

$$\left\{[f, 1], [f, x_i], [f, \xi_i], [f, f_j] \mid i \in R, j = 1, 2\right\}$$

are linearly independent, which means that $I(\mathrm{ad} f) > s$. The proof is complete. □

Lemma 3.5 ([22]). *Let $f_i = g_i + h_i$, where $f_i, g_i, h_i \in \Omega,\ i = 1, 2, \ldots, t$. If $\{g_i \mid i = 1, 2, \ldots, t\}$ are linearly independent and $\mathrm{span}_{\mathbb{F}}\{g_i \mid i = 1, 2, \ldots, t\} \cap \mathrm{span}_{\mathbb{F}}\{h_i \mid i = 1, 2, \ldots, t\} = 0$. Then $\{f_i \mid i = 1, 2, \ldots, t\}$ are linearly independent.*

Theorem 3.6. *$I(\mathrm{Der}(L)) = s$. If $\varphi \in h(\mathrm{Der}\,(L))$, then $I(\varphi) = s$ if and only if $0 \neq \varphi \in \langle B\rangle$.*

Proof. According to Lemma 3.1, we can suppose that $I(h(\mathrm{Der}\,(\Omega))) \leq s$. Let $\varphi \in h(\mathrm{Der}(\Omega))$ and $I(\varphi) \leq s$. By virtue of Proposition 2.1, we may assume that

$$\varphi = \mathrm{ad}\, f + \sum_{i\in M}\sum_{v=1}^{s_i} \alpha_{iv} D_i^{p^v} + D_\theta,$$

where $f \in \Omega$, $\alpha_{iv} \in \mathbb{F}$. We shall prove that $\alpha_{iv} = 0$ and $\theta = 0$. Suppose that there is a $l \in M$ such that $\alpha_{lv} \neq 0$. Put $t = \max\{v \mid \alpha_{lv} \neq 0\}$. Let

$$U = \{k \in Q \mid k_l = p^t, p^{s_i} \leq k_i \leq \pi_i, \forall i \in M\backslash\{l\}\}.$$

For any $k \in U$, we have

$$\varphi(x^k y^\lambda \xi^u) = \gamma x^{k-p^t e_l} y^\lambda \xi^u + g,$$

where $\gamma \in \mathbb{F}$ and g is indeed a $\mathbb{F}$-linear combination of some elements of $\{x^{k'} y^\eta \xi^\delta \mid k'_l \neq 0\}$. It follows from Lemma 3.5 that

$$\{\gamma x^{k-p^t e_l} y^\lambda \xi^u + g \mid k \in U, \lambda \in H, u \in \mathbb{B}(q)\}$$

are linearly independent. Then $I(\varphi) \geq (p-1)^{r-2} p^m 2^q > s$, contradicting $I(\varphi) \leq s$. So $\alpha_{iv} = 0$. Now $\varphi = \mathrm{ad} f + D_\theta$. Put $\varepsilon(f) = h$. If $\mathrm{zd}(h) \leq -1$, then

$$\varepsilon(\varphi(z)) = \varepsilon(\mathrm{ad} h(z)), \quad \forall z \in \Omega.$$

By virtue of Lemma 3.4, we see that $I(\mathrm{ad}(h)) > s$. Then $I(\varphi) > s$. It is a contradiction. Hence $\mathrm{zd}(h) \geq 0$. Suppose that $\theta \neq 0$. Then there is some $\eta \in H$ such that $\theta(\eta) \neq 0$. If $\mathrm{zd}(h) \geq 1$, we set

$$U_1 = \left\{x^k y^\eta \xi^u \mid 2k_1 + \sum_{i\in M_1} k_i + |u| = 2, \theta(\eta) \neq 0\right\}.$$

Then

$$\varepsilon(\varphi(z)) = \varepsilon(\mathrm{ad}\, h(z) + D_\theta(z)) = \theta(\eta) z, \quad \forall z \in U_1.$$

So $\{\varepsilon(\varphi(z)) \mid z \in U_1\}$ are linearly independent. Thus $I(\varphi) > s$, a contradiction. Let $\mathrm{zd}(h) = 0$. Set

$$h = \left(\sum_{i,j\in M_1} a_{ij} x_i x_j + \sum_{i\in M_1, j\in T} b_{ij} x_i \xi_j + \sum_{i,j\in T} c_{ij} \xi_i \xi_j + \mu x_1\right) y^\lambda,$$

where $a_{ij}, b_{ij}, c_{ij}, \mu \in \mathbb{F}$. Put

$$\begin{aligned} U_2 = &\left\{\prod_{j=1}^{t} \xi_{r+j} y^\eta \mid t = 1, \ldots, q\right\} \\ &\cup \{x^{e_i+e_{i'}} y^\eta \xi^\omega,\ x^{2e_i+2e_{i'}} y^\eta \xi^\omega \mid i = 2, \ldots, n+1\} \\ &\cup \{x^{3e_i+3e_{i'}} y^\eta \xi^\omega,\ x^{4e_i+4e_{i'}} y^\eta \xi^\omega \mid i = 2, \ldots, n+1\}. \end{aligned}$$

By direct computation , we have

$$\varepsilon(\varphi(z))) = (\mathrm{ad} h + D_\theta)(z) \neq 0, \quad \forall z \in U_2.$$

Considering the $\mathbb{Z}$-degree of $\varepsilon(\varphi(z))$, we obtain that $\{\varepsilon(\varphi(z)) \mid z \in U_2\}$ are linearly independent. So $I(\varphi) \geq 4n + q > s$, a contradiction; that is, $\theta = 0$. It follows that $\varphi = \mathrm{ad} f$. Lemma 3.4 implies that $I(\mathrm{Der}(\Omega)) = s$ and if $I(\varphi) = s$, then $\varphi = \mathrm{ad}\, x^\pi \xi^\omega \alpha(y)$. Assume that $\alpha(y) \notin \langle \chi(y)\rangle$. Since $\langle \chi(y)\rangle$ is the only one-dimensional ideal of $\mathbb{F}[y]$ (see [17]), there is a $\nu \in H$ such that $\alpha(y)$ and $\alpha(y) y^\nu$ are linearly independent. Now $\varphi(y^\nu) = [x^\pi \xi^\omega \alpha(y), y^\nu] = (\nu - 1) x^{\pi - e_1} \xi^\omega \alpha(y) y^\nu$ implies that the images of the s+1 elements $1, y^\nu, x_i, \xi_j, \forall i \in M, \forall j \in T$ are linearly independent. So $I(\varphi) > s$. It contradicts the fact that $I(\varphi) = s$. Therefore $\alpha(y) \in \langle \chi(y)\rangle$ and $\varphi \in \langle B\rangle$. □

Let ρ be the induced representation of $\mathfrak{C}$ on $\Omega/\mathfrak{C}$, i.e.,

$$\rho(f):\ (g+\mathfrak{C})\mapsto [f,g]+\mathfrak{C},\quad \text{where } f\in\mathfrak{C}, g\in\Omega.$$

Lemma 3.7. (1) *ρ is irreducible.*
(2) *$\mathfrak{C}$ is an invariant maximal subalgebra of Ω.*

Proof. (1) For all $f\in\Omega$, the element $f+\mathfrak{C}\in\Omega/\mathfrak{C}$ will be denoted by $\bar{f}$. Assume that V is a nonzero submodule of $\Omega/\mathfrak{C}$ and

$$0\neq\bar{f}=\gamma\,\bar{1}+\delta\overline{x_1}+\sum_{i\in M_1}\alpha_i\overline{x_i}+\sum_{j\in T}\beta_j\overline{\xi_j}\in V,$$

where $\gamma,\delta,\alpha_i,\beta_j\in\mathbb{F}$. If there is some $i\in M_1$ (or $j\in T$) such that $\alpha_i\neq 0$ (or $\beta_j\neq 0$), then

$$\rho(x_ix_{i'})\bar{f}=\left[x_ix_{i'},\sum_{i\in M_1}\alpha_ix_i\right]+\mathfrak{C}=[i']\alpha_i\overline{x_i}\in V\quad\left(\text{or } \rho(\xi_i\xi_j)\bar{f}=\left[\xi_i\xi_j,\sum_{j\in T}\beta_j\xi_j\right]+\mathfrak{C}=\beta_j\overline{\xi_i}\in V.\right)$$

If $\alpha_i=\beta_j=0$, $\forall i\in M_1,\forall j\in T$, when $\gamma\neq 0$, we obtain

$$\rho(x_1x_i)\bar{f}=[x_1x_i,\gamma]+[x_1x_i,\delta x_1]+\mathfrak{C}=\gamma\overline{x_i}\in V\quad(\text{or } \rho(x_1\xi_j)\bar{f}=\gamma\overline{\xi_j}\in V).$$

When $\gamma=0$, we have $\delta\neq 0$ and then

$$\rho(x_i(1-y^\lambda))\bar{f}=[x_i(1-y^\lambda),\delta x_1]+\mathfrak{C}=-\delta((1-\mu_i)-(1-\mu_i)y^\lambda)x_i-\delta\lambda x_iy^\lambda+\mathfrak{C}=-\delta\lambda\overline{x_iy^\lambda}\in V;$$

that is, $\overline{x_i}=\overline{x_iy^\lambda}\in V$. Similarly, $\overline{\xi_j}=\overline{\xi_jy^\lambda}\in V$. In all cases we have $\overline{x_i}\in V$ (or $\overline{\xi_j}\in V$) for some $i\in M_1$ (for some $j\in T$). So

$$\begin{aligned}[i']\rho\big(\lambda^{-1}(1-y^\lambda)x_{i'}\big)\overline{x_i}&=[i']\lambda^{-1}[(1-y^\lambda)x_{i'},x_i]+\mathfrak{C}=\lambda^{-1}(1-y^\lambda)+\mathfrak{C}\equiv 1+\mathfrak{C}=\bar{1}\in V\\ &\big(\text{or }-\rho\big(\lambda^{-1}(1-y^\lambda)\xi_j\big)\overline{\xi_j}=\bar{1}\in V\big).\end{aligned}$$

Then $\overline{x_1}=\rho(2^{-1}x_1^2)(\bar{1})\in V$, $\overline{x_i}=\rho(x_1x_i)(\bar{1})\in V$, $\forall\, i\in M_1$, and $\overline{\xi_j}=\rho(x_1\xi_j)(\bar{1})\in V$, $\forall j\in T$. It follows that $V=\Omega/\mathfrak{C}$.

(2) $\mathfrak{C}$ is invariant according to Lemma 3.1 and Theorem 3.6. Let L be any subalgebra containing $\mathfrak{C}$, then $L/\mathfrak{C}$ is a submodule of $\Omega/\mathfrak{C}$. By the first part of this Lemma, $L=\Omega$ or $L=\mathfrak{C}$ and thereby $\mathfrak{C}$ is maximal. □

Recall that $\Omega=\Omega(r,m,q,\underline{s},H)$. Set $\mathfrak{B}=\langle B\rangle=\langle x^\pi\xi^\omega\chi(y)\rangle$. Let $\Omega'=\Omega(r',m',q',\underline{s}',H')$. Put $\pi'=(\pi'_1,\ldots,\pi'_{r-1})$, $\omega'=\langle r'+1,\ldots,r'+q'\rangle$. Set $\mathfrak{B}'=\langle x^{\pi'}\xi^{\omega'}\chi'(y)\rangle$.

Similarly, Ω' possesses a filtration: $\{\Omega'_{(i)}\mid i\geq -1\}$, where $\Omega'_{(-1)}=\Omega'$, $\mathfrak{C}'=\Omega'_{(0)}$ and

$$\Omega'_{(i)}=\{f\in\Omega'_{(i-1)}\mid[f,\Omega'_{(-1)}]\subseteq\Omega'_{(i-1)}\},\quad\forall\, i\geq 1,\tag{6}$$

Lemma 3.8. *If σ is an isomorphism of Ω onto Ω', then $\sigma(\Omega_{(0)})=\Omega'_{(0)}$.*

Proof. From Lemmas 3.4 and 3.6, we see that $\sigma(\mathfrak{B})=\mathfrak{B}'$. Since

$$\begin{aligned}&[f,\mathfrak{B}]=0\iff[\sigma(f),\sigma(\mathfrak{B})]=0,\quad\forall f\in\Omega,\\ \sigma(\Omega_{(0)})&=\sigma(\mathfrak{C})=\sigma\big\{f\in\Omega\mid[f,\mathfrak{B}]=0\big\}=\big\{\sigma(f)\in\Omega'\mid[f,\mathfrak{B}]=0\big\}\\ &=\big\{\sigma(f)\in\Omega'\mid[\sigma(f),\sigma(\mathfrak{B})]=0\big\}=\big\{g\in\Omega'\mid[g,\mathfrak{B}']=0\big\}=\mathfrak{C}'=\Omega'_{(0)}.\quad\square\end{aligned}$$

By virtue of equalities (2), (6) and Lemma 3.8, we obtain the following Proposition.

Proposition 3.9. *Let σ be an isomorphism of Ω onto Ω'. Then $\sigma(\Omega_{(i)})=\Omega'_{(i)}$, $i\geq -1$.*

Theorem 3.10. *The filtration of Ω is invariant under the automorphism group of Ω.*

Proof. This is a direct consequence of Proposition 3.9. □

Theorem 3.11. $\Omega(r,m,q,\underline{s},H)\cong\Omega'(r',m',q',\underline{s}',H')\iff r=r', m=m', q=q', s_1=s'_1$ *and*

$$\big\{\{s_2,s_{2'}\},\ldots,\{s_{(n+1)},s_{(n+1)'}\}\big\}=\big\{\{s'_2,s'_{2'}\},\ldots,\{s'_{(n+1)},s'_{(n+1)'}\}\big\}.\tag{7}$$

Proof. The sufficient condition is obvious. We shall prove the necessary condition. As $\dim \Omega = \dim \Omega'$, i.e., $2^q p^{\sum_{i\in M}(s_i+1)+m} = 2^{q'} p^{\sum_{i\in M'}(s_i'+1)+m'}$. So $q = q'$. If σ is an isomorphism of Ω onto Ω' and $D \in \operatorname{Der}\Omega$, then the mapping $D \mapsto \sigma D \sigma^{-1}$ is an isomorphism of $\operatorname{Der}\Omega$ onto $\operatorname{Der}\Omega'$, i.e., $\operatorname{Der}\Omega \cong \operatorname{Der}\Omega'$. Hence $I(\operatorname{Der}\Omega) = I(\operatorname{Der}\Omega')$; that is, $r+q = r'+q'$. Thus $r = r'$. Moreover the outer derivation subspace has the same dimension, since the outer derivation D_θ is not ad −nilpotent, $m = m'$.

Note that $\Omega = \mathfrak{C} \oplus \mathfrak{N}$ and $\Omega' = \mathfrak{C}' \oplus \mathfrak{N}'$. One may easily verify that $\sigma(\mathfrak{N}) = \mathfrak{N}'$ by Lemma 3.8. Recall that $\sigma(\Omega_\alpha) = {\Omega'}_\alpha$, where $\alpha \in \mathbb{Z}_2$. Put

$$V_i = \{f \in \Omega_{(i)} \cap \Omega_{\bar{0}} \mid \operatorname{ad} f\ (\mathfrak{N} \cap \Omega_{\bar{1}}) = 0\}, \quad i \geq -1, \tag{8}$$

$$V_i' = \{g \in \Omega'_{(i)} \cap \Omega'_{\bar{0}} \mid \operatorname{ad} g\ (\mathfrak{N}' \cap \Omega'_{\bar{1}}) = 0\}, \quad i \geq -1. \tag{9}$$

Then $V_i = \Omega(r, m, \underline{s}, H)_{(i)}$ and $V_i' = \Omega(r, m, \underline{s}', H')_{(i)}$. Let $V = \bigcup_{i\geq -1} V_i$ and $V' = \bigcup_{i\geq -1} V_i'$. It is easy to show that $V = \Omega(r, m, \underline{s}, H)$ and $V' = \Omega(r, m, \underline{s}', H')$. It follows from (8) and (9) that $\sigma(V_i) = V_i'$, $\forall i \geq -1$. Hence $\sigma(V) = V'$. Therefore $\Omega(r, m, \underline{s}, H) \cong \Omega(r, m, \underline{s}', H')$. By the consequence of Lie algebra (see [3]), we obtain $s_1 = s_1'$ and equality (7) holds. □

By a slightly refined discussion, similar results can be obtained for Ω^*.

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